Coherent State Path Integrals and Vacuum Structure in Thermal ϕ^4 Theory on a Finite Volume

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Abstract. We present a derivation of the thermal partition function in scalar thermal field theories with an emphasis on finite-size effects. By carefully transitioning from the Hamiltonian operator formalism to the path integral using coherent states, we obtain corrections to the partition function that expose vacuum structures. These corrections are rarely discussed in previous literature.

1 Introduction

Understanding the behavior of matter under extreme conditions — such as those present in the early universe or recreated in heavy-ion collisions — is a central goal of high-energy physics. In particular, the study of the Quark-Gluon Plasma (QGP) provides deep insights into the early universe and the dynamics of strongly interacting matter. A powerful framework for investigating such systems is *Thermal Field Theory* (TFT), which extends Quantum Field Theory (QFT) to finite temperature and density. TFT is widely used in high-energy physics, nuclear physics, and cosmology, and plays a crucial role in describing phenomena such as cosmological phase transitions, thermal particle production, and QGP formation in heavy-ion collisions at RHIC (Brookhaven National Lab) and the LHC (CERN). It is also an essential tool in lattice QCD studies of the QCD equation of state and in modeling dense astrophysical systems such as neutron stars and their mergers. Recent applications of TFT include finite-size effects and Casimir physics at finite temperature [1, 2, 3, 4]. For comprehensive reviews of TFT, see e.g. [5, 6, 7, 8, 9].

A key quantity in TFT is the thermal partition function, which encodes the equilibrium statistical properties of quantum fields. Here, we focus on the path integral formulation of the partition function in a scalar field theory and present recent results obtained from a careful construction of the path integral using coherent states [10]. The standard treatments in TFT derive the path integral representation of the partition function by inserting decompositions of unity $\mathbb{1} \equiv \int d\phi\, |\phi\rangle\langle\phi|$ and $\mathbb{1} \equiv \int d\pi\, |\pi\rangle\langle\pi|$ using eigenstates of the field operator and its canonically conjugate momentum, $\hat{\phi}\,|\phi\rangle = \phi\,|\phi\rangle$, $\hat{\pi}\,|\pi\rangle = \pi\,|\pi\rangle$. While the use of these unities makes the derivation of the path integral straightforward, it is unclear how these eigenkets and, especially, the integrals over them, are defined. In our derivation, we rather use well-defined coherent states.

Of special mention in our derivation is the connection between coherent states in the field basis with coherent states in the ladder basis; deferring the details for now,

$$\begin{aligned} |\boldsymbol{\alpha}(\tau,t)\rangle &\sim e^{i\int d^n x\, \phi(\tau,\mathbf{x})\hat{\pi}(t,\mathbf{x}) - \pi(\tau,\mathbf{x})\hat{\phi}(t,\mathbf{x})}\,|0\rangle \\ &\sim e^{\int \frac{d^n p}{(2\pi)^n 2E_{\mathbf{p}}}\left(\alpha(\tau,\mathbf{p})\hat{a}^\dagger(t,\mathbf{p}) - \alpha^*(\tau,\mathbf{p})\hat{a}(t,\mathbf{p})\right)}\,|0\rangle\,, \end{aligned} \tag{1}$$

where

$$\hat{a}(t, \mathbf{p}) \, |\boldsymbol{\alpha}(\tau, t)\rangle = \alpha(\tau, \mathbf{p}) \, |\boldsymbol{\alpha}(\tau, t)\rangle \, . \tag{2}$$

Through this derivation, we show that the vacuum energy contribution to the action arises naturally in the path integral of the free theory when the Hamiltonian is expressed in terms of ladder operators. We then extend the use of coherent states in the field-operator basis to the interacting theory. For the interacting case, we find, alongside the vacuum energy, a novel contribution to the action: a vacuum bubble that acts as a mass correction.

2 Motivation

The final goal that motivated this research was to analytically compute next-to-leading order (NLO) finite-size corrections to the QCD equation of state. Understanding these corrections is of growing importance across a wide range of physical systems, as discussed in the introduction 1, where the thermodynamic limit is applicable.

The reason we are interested in the influence of finite system size on the thermodynamic and statistical properties of the QGP is mainly because this is a largely unexplored area of research. The Casimir effect, predicted by Dutch physicist Hendrik Casimir for electromagnetic systems in 1948 [11], provides a striking example of how confinement in finite volumes can induce measurable modifications to physical observables. Figure 1 shows experimental measurements from [12], illustrating that Casimir forces are detectable at length scales on the order of $200\,\mathrm{nm}$. While the effect is relatively small at such distances, its magnitude increases significantly as the system size decreases, becoming comparable to atmospheric pressures around $10\,\mathrm{nm}$.

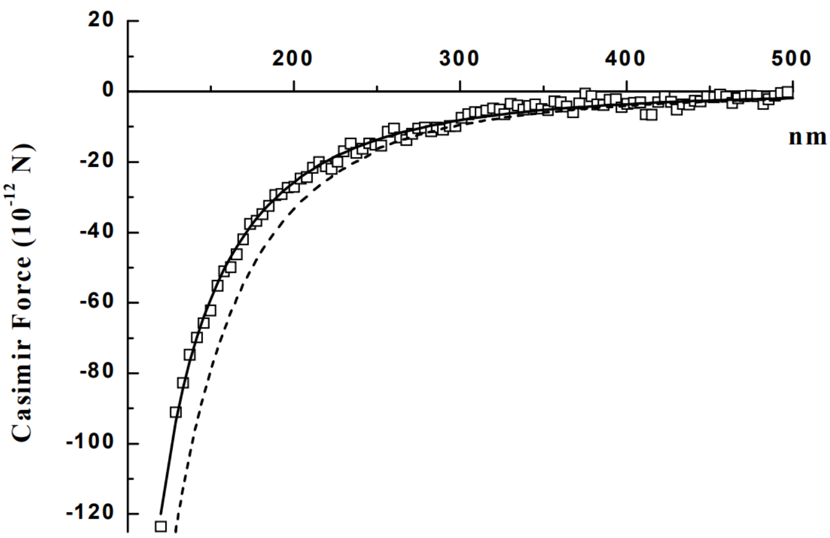

Figure 1: Casimir Force plot. Figure adapted from [12].

Given that the QGP formed in heavy-ion collisions occupies volumes on the scale of a nucleus, these finite-size corrections to thermodynamic quantities such as pressure and energy density may be significant. For example, [2] shows that even in a simple, non-interacting massless scalar field theory, finite volume effects induce substantial deviations in thermodynamic observables. Such corrections could alter the QGP equation of state and, consequently, affect key transport properties like η/s. Importantly, the sign and magnitude of these finite volume effects remain largely unexplored, raising the possibility that the QGP might exhibit different fluidity characteristics at finite system size than those inferred from infinite volume approximations. This motivates a systematic study of finite size effects on the thermodynamics and hydrodynamics of the QGP.

The path toward this ambitious goal involves several intermediate steps. As a first step, we have computed the partition function of an interacting ϕ^4 theory in a finite volume at finite temperature. Subsequent steps include computing the equation of state and the trace anomaly for a range of interacting quantum field theories in finite volumes. Starting with scalar theories such as ϕ^4, we will gradually advance to more complex systems, including fermionic theories, spin systems relevant to condensed matter physics, and ultimately, QCD. Each step serves both as a validation of our methods and as a bridge toward applications in realistic settings.

The relevance of these results extends to numerical simulations as well. In particular, the analytical finite-size corrections computed here can serve as benchmarks and guiding insights for Lattice QCD simulations, where volume effects must be carefully controlled and interpreted. More broadly, this line of research aims to enhance our theoretical understanding of quantum fields in confined geometries, offering deeper analytical control over models that underpin many frontier topics in high-energy and condensed matter physics.

3 Results

All derivations in this section are performed in discretized spacetime. However, for readability, we present the results directly in their continuum-limit form. A number of technical subtleties are omitted for brevity. For a more detailed and rigorous treatment, we refer the reader to Ref. [10].

3.1 Free Theory Partition Function

For the sake of simplicity, we start to examine a free massive single component scalar field theory in $n + 1$ dimensions. The weak $\lambda\phi^4$ interactions for this classical real scalar theory will follow later on. Due to the thermal system, the discretization even in equilibrium is non-trivial. The Lagrangian for the considered theory[1] is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \qquad (3)$$
$$\phi^*(t,\mathbf{x}) = \phi(t,\mathbf{x}).$$

In our scalar field theory, we assume that the vacuum expectation value of the scalar field is zero, $\langle\phi(t,\mathbf{x})\rangle_0 = 0$, as is typical for scalar theories in thermal equilibrium without spontaneous symmetry breaking.

The Hamiltonian is obtained via a Legendre transform,

$$\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L}, \quad \pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$
$$\therefore \mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\cdot\phi)^2 + m^2\phi^2\right).$$

In order to construct the path integral formulation function of the partition function in a controlled way, we start with the trace over the thermal density matrix,

$$Z = \mathrm{Tr}\{e^{-\beta H}\}, \qquad (4)$$

where $\beta = \frac{1}{T}$.

Applying trotterization [13], we divide the partition function in eq. 4 into $N \to \infty$ segments and insert complete sets of coherent states in between each segment to find the thermal trace trotter formula (for further details, see Ref. [10])

$$Z = \prod_{i=0}^{N} N_i \int d\alpha_i(\mathbf{p}) d\alpha_i^*(\mathbf{p}) \langle\boldsymbol{\alpha}_N| e^{-\frac{\beta}{N}\hat{H}} |\boldsymbol{\alpha}_{N-1}\rangle \langle\boldsymbol{\alpha}_{N-1}| e^{-\frac{\beta}{N}\hat{H}} |\boldsymbol{\alpha}_{N-2}\rangle \cdots \langle\boldsymbol{\alpha}_1| e^{-\frac{\beta}{N}\hat{H}} |\boldsymbol{\alpha}_0\rangle. \qquad (5)$$

By evaluating the matrix elements in Eq. 5, and performing a well-motivated change of variables followed by a Fourier transformation, we obtain a path integral over field configurations weighted by the exponentiated Euclidean action. A careful treatment of this procedure reveals that the vacuum energy contribution emerges naturally as part of the path integral. Although the full derivation involves several subtleties, the resulting path integral expression for the thermal partition function in the free theory takes the form

$$Z = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right]$$
$$= \int D\phi\, e^{-S^E + \langle\hat{H}\rangle_0}, \qquad (6)$$

where

$$S^E = \int_0^\beta d\tau \int d^n\mathbf{x}\frac{1}{2}\left[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right]. \qquad (7)$$

[1] We work with the mostly-minus metric in $D = n + 1$ Minkowskian spacetime; $\eta^M_{\mu\nu} = \mathrm{diag}(+1, -1, \cdots, -1)$, use natural units throughout this work, $\hbar = c = k_B = 1$, and denote $\mathrm{i} \equiv \sqrt{-1}$.

3.2 *Interacting ϕ^4 Theory Partition Function*

When interactions are included, we will show that some vacuum expectation values (VEV) arise that couple to the other terms in the Lagrangian and modify the structure of the counterterms. We start with the bare scalar ϕ^4 Lagrangian in $D = n + 1$ dimensions and with the coupling constant λ_0,

$$\mathcal{L}_0 \equiv \frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4, \tag{8}$$

and renormalize using multiplicative renormalization. We define

$$\begin{aligned}
\phi_0 &\equiv \sqrt{Z_\phi}\phi_r, \quad Z_\phi \equiv 1 + \delta_\phi,\\
m_0 &\equiv \frac{\sqrt{Z_m}}{\sqrt{Z_\phi}}m_r, \quad Z_m \equiv 1 + \frac{\delta_m}{m_r},\\
\lambda_0 &\equiv \frac{Z_\lambda}{Z_\phi^2}\lambda_r, \quad Z_\lambda \equiv 1 + \frac{\delta_\lambda}{\lambda_r}.
\end{aligned} \tag{9}$$

Inserting the renormalized parameters from eq. 9 into the bare Lagrangian, eq. 8 yields

$$\begin{aligned}
\mathcal{L}_r &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}},\\
\mathcal{L}_{\text{free}} &= \frac{1}{2}\partial_\mu\phi_r\,\partial^\mu\phi_r - \frac{1}{2}m_r^2\phi_r^2,\\
\mathcal{L}_{\text{int}} &= -\frac{\lambda_r}{4!}\phi_r^4 + \frac{1}{2}\delta_\phi\partial_\mu\phi_r\,\partial^\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4.
\end{aligned} \tag{10}$$

Performing a Legendre transform then gives the renormalized Hamiltonian,

$$\begin{aligned}
\mathcal{H}_r =&\mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}},\\
\mathcal{H}_{\text{free}} =&\frac{1}{2}\pi_r^2 + \frac{1}{2}(\nabla\cdot\phi_r)^2 + \frac{1}{2}m_r^2\phi_r^2,\\
\mathcal{H}_{\text{int}} =&\frac{\lambda_r}{4!}\phi_r^4 + \frac{1}{2}\delta_\phi\left(\pi_r^2 + \frac{1}{2}(\nabla\cdot\phi_r)^2\right) + \frac{1}{2}\delta_m\phi_r^2\\
&+ \frac{\delta_\lambda}{4!}\phi_r^4.
\end{aligned} \tag{11}$$

Going through the same procedure as for the free theory, we start off with the trace over the thermal density matrix of the renormalized ϕ^4 theory, trotterize, insert complete sets of coherent states and compute the matrix elements. This process reveils the following form of the path integral partition function of the renormalized interacting ϕ^4 theory:

$$\begin{aligned}
Z &= \text{Tr}\left[e^{-\beta\hat{H}_r}\right]\\
&= \int D\phi\, e^{-S_r^E + \frac{\lambda_r+\delta_\lambda}{4}\langle\hat{\phi}^2\rangle_0\phi^2 + \langle\hat{H}_r\rangle_0},
\end{aligned} \tag{12}$$

where

$$S_r^E = \int_0^\beta d\tau \int d^n\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\nabla\cdot\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_m\phi_r^2 + \frac{\delta_\lambda}{4!}\phi_r^4\right]. \tag{13}$$

In the path integral of eq. 12, we have found the usual Euclidean action, the VEV of the Hamiltonian, but notable exception from the literature arises from the coupling between the VEV $\langle 0|\,\hat{\phi}^2\,|0\rangle$ and the classical field configurations ϕ^2. This coupling introduces a nontrivial correction to the mass counterterm and marks a key distinction from the standard path integral formulations in the literature where such couplings between operator VEVs and classical fields are absent.

4 Conclusions and Outlook

In this work, we systematically derived the path integral formulation of the thermal partition function for a scalar quantum field theory from first principles. We began by constructing the partition function for a free scalar field theory using well-defined field-theoretic coherent states. In contrast to the ill-defined resolutions of unity used in the standard literature, based on eigenstates of field operators, we employed rigorously defined coherent states to formulate the path integral.

We provided an explicit connection between coherent states in the ladder operator basis and those in the field operator basis. This mapping is crucial for deriving the path integral formulation of the interacting theory partition function and, to our knowledge, has not been presented elsewhere. In both the ladder operator basis, and the field operator basis, the vacuum energy emerges explicitly through the matrix element of the Hamiltonian with coherent states. In both approaches, we obtained an identical, well-defined path integral representation of the partition function as a finite product of integrals over real numbers. This includes the familiar Euclidean action as well as an explicit vacuum expectation value (VEV) of the Hamiltonian, absent from traditional treatments.

We then extended our formalism to the interacting ϕ^4 theory using the field-basis coherent states. In this case, we found two novel contributions in the exponent of the partition function: (i) the VEV of the full Hamiltonian and (ii) a second VEV coupling the vacuum bubble $\langle 0|\hat{\phi}^2|0\rangle$ to the mass term ϕ^2. These terms are not present in any standard derivation and emerge before integration over conjugate momenta, pointing to their structural origin in the path integral formulation.

Given that traditional treatments miss these VEVs, we are led to question the validity of assuming a resolution of the identity via $\mathbb{1} = \int d\phi\, |\phi\rangle\langle\phi|$ and $\mathbb{1} = \int d\pi\, |\pi\rangle\langle\pi|$. In particular, we have found no rigorous construction in the literature for defining the identity operator in terms of field eigenstates. We suspect this assumption is the root cause of the discrepancy.

Whether these additional VEVs have physical consequences remains an open and intriguing question. Consider first the VEV of the Hamiltonian. While often regarded as physically irrelevant, this term plays a central role in the Casimir effect, where energy differences between confined and infinite-volume vacua manifest as measurable forces. Additionally, in gravitational theories, this VEV could contribute to an effective cosmological constant.

The second VEV, which couples to the mass term, implies that the effective mass depends on the system's finite size. This could have significant consequences for physical observables, especially in systems with spontaneous symmetry breaking (SSB). In such scenarios, even more vacuum diagrams could couple to the theory: the tadpole, for example, might couple to a ϕ^3 term, and the three-point (Mercedes) diagram to a ϕ term. Furthermore, the sign of the vacuum correction is not guaranteed, if negative and sufficiently large, it could induce SSB even if the bare mass is positive. Such a shift would impact phase structure, thermodynamic quantities like pressure, energy and entropy density, the speed of sound, and the trace anomaly.

Looking forward, our approach provides a concrete foundation for computing higher-order corrections to thermodynamic observables in both infinite and finite volumes. This enables systematic investigations of how finite-size effects modify the equation of state and trace anomaly in scalar field theories [3, 4]. These studies offer an ideal testbed for later extensions to fermions and gauge fields. In particular, carrying out an analogous construction in Quantum Electrodynamics (QED), and ultimately in Quantum Chromodynamics (QCD), would allow for nonperturbative evaluation of finite-volume corrections to the QCD equation of state. Such corrections could have significant implications for lattice QCD simulations [14, 15, 16] and for the interpretation of signals of quark-gluon plasma formation in small systems. They may also impact observables sensitive to conformal symmetry breaking [17, 18], such as flow in small heavy-ion collisions [19] or energy-energy correlators [20, 21, 22, 23], which are increasingly being explored as precision probes of the QGP.

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References

[1] I. Ghisoiu, "Finite size effects in quantum field theory," Master's thesis, Bielefeld U., 2010.

[2] S. Mogliacci, I. Kolbé, and W. A. Horowitz, "Geometrically confined thermal field theory: Finite size corrections and phase transitions," *Phys. Rev. D*, vol. 102, no. 11, p. 116017, 2020.

[3] W. Horowitz and A. Rothkopf, "The QCD Equation of State in Small Systems," *SciPost Phys. Proc.*, vol. 10, p. 025, 2022.

[4] W. A. Horowitz and J. F. D. Plessis, "Finite system size correction to NLO scattering in $\phi4$ theory," *Phys. Rev. D*, vol. 105, no. 9, p. L091901, 2022.

[5] N. P. Landsman and C. G. van Weert, "Real and Imaginary Time Field Theory at Finite Temperature and Density," *Phys. Rept.*, vol. 145, p. 141, 1987.

[6] J. I. Kapusta and C. Gale, *Finite-temperature field theory: Principles and applications*, ser. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011.

[7] M. Laine and A. Vuorinen, *Basics of Thermal Field Theory: A Modern Primer*, ser. Lecture Notes in Physics. Cham, Switzerland: Springer, 2016, vol. 925.

[8] M. Strickland, *Relativistic Quantum Field Theory, Volume 3*. IOP, 11 2019.

[9] M. G. Mustafa, "An introduction to thermal field theory and some of its applications," 2022, arXiv preprint: 2207.00534. [Online]. Available: https://arxiv.org/abs/2207.00534

[10] R. Roosenstein, M. Attems, and W. A. Horowitz, "Coherent State Path Integral Reveals Unexpected Vacuum Structure in Thermal Field Theory," 7 2025.

[11] H. B. G. Casimir, "On the attraction between two perfectly conducting plates," *Proc. K. Ned. Akad. Wet.*, vol. 51, pp. 793–795, 1948.

[12] G. L. Klimchitskaya and V. M. Mostepanenko, "Experiment and theory in the Casimir effect," *Contemp. Phys.*, vol. 47, pp. 131–144, 2006.

[13] H. F. Trotter, "On the product of semi-groups of operators," *Proceedings of the American Mathematical Society*, vol. 10, no. 4, pp. 545–551, 1959.

[14] T. Umeda, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, and H. Ohno, "Fixed scale approach to equation of state in lattice qcd," *Phys. Rev. D*, vol. 79, p. 051501, 2009.

[15] A. Bazavov, T. Bhattacharya, M. Cheng, and P. Petreczky, "Equation of state and qcd transition at finite temperature," *Phys. Rev. D*, vol. 80, p. 014504, 2009.

[16] P. Huovinen and P. Petreczky, "Qcd equation of state and hadron resonance gas," *Nucl. Phys. A*, vol. 837, p. 26, 2010.

[17] A. V. Belitsky, "QCD evolution equations," in *The Phenomenology of Large N(c) QCD*, 4 2002, pp. 128–142.

[18] M. Attems, J. Casalderrey-Solana, D. Mateos, I. Papadimitriou, D. Santos-Oliván, C. F. Sopuerta, M. Triana, and M. Zilhão, "Thermodynamics, transport and relaxation in non-conformal theories," *JHEP*, vol. 10, p. 155, 2016.

[19] L. Yan, "A flow paradigm in heavy-ion collisions," *Int. J. Mod. Phys. A*, vol. 32, no. 14, p. 1730005, 2017.

[20] D. M. Hofman and J. Maldacena, "Conformal collider physics: Energy and charge correlations," *JHEP*, vol. 05, p. 012, 2008.

[21] H. Chen, I. Moult, X. Zhang, and H. X. Zhu, "Rethinking jets with energy correlators: Tracks, resummation, and analytic continuation," *Phys. Rev. D*, vol. 102, no. 5, p. 054012, 2020.

[22] C. Andres, F. Dominguez, R. Kunnawalkam Elayavalli, J. Holguin, C. Marquet, and I. Moult, "Resolving the Scales of the Quark-Gluon Plasma with Energy Correlators," *Phys. Rev. Lett.*, vol. 130, no. 26, p. 262301, 2023.

[23] A. Budhraja, H. Chen, and W. J. Waalewijn, "ν-point energy correletors with FastEEC: Small-x physics from LHC jets," *Phys. Lett. B*, vol. 861, p. 139239, 2025.