Piston-Driven Shock Wave Test Problem for Validating Magnetohydrodynamic Models in Astrophysics

Magdeline M. Seabi [1], Azwinndini Muronga[2]

[1]Department of Physics, Nelson Mandela University, South Campus, University Way, Summerstrand, Gqeberhah, South Africa, 6001
[2]Nelson Mandela University, South Campus, University Way, Summerstrand, Gqeberhah, South Africa, 6001

E-mail: s226052184@mandela.ac.za,magdeline.seabi@nithecs.ac.za

Abstract. Understanding matter under extreme conditions is critical for both astrophysical and high-energy physics applications. This study presents a hydrodynamic piston-driven shock model, developed to generate analytical benchmark solutions for shock propagation in core-collapse supernovae environments. The model captures key features of compressible flow, including forward shock formation and rarefaction waves, based on the Rankine–Hugoniot jump conditions. In parallel, the Brio–Wu relativistic MHD shock tube is simulated using the PLUTO code to validate its ability to resolve complex wave structures in magnetised plasmas. Together, these two models provide complementary tools for verifying numerical shock solvers and establishing a foundation for future extensions involving relativistic magnetohydrodynamics (RMHD).

1 Introduction

In both astrophysical environments and relativistic heavy-ion collisions (HIC), matter exists under conditions of extreme temperature, density, and magnetic field strength. Understanding the behaviour of matter in these regimes is crucial for interpreting observations from supernovae, neutron star mergers, and experimental results from particle colliders such as the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) [1, 2]. In such systems, the interplay between shock waves, fluid dynamics, and magnetic fields gives rise to complex behaviours that challenge existing numerical models.

One of the most critical processes in core-collapse supernovae (CCSNe) is the formation and outward propagation of a shock wave during the so-called "bounce" phase. As the core of a massive star collapses under gravity, it reaches nuclear densities, triggering a rebound that launches a shock wave into the surrounding stellar material. This shock is initially strong but can stall due to energy losses from neutrino emission and nuclear dissociation. Accurately capturing the physics of this shock, its interaction with infalling material, the effects of pressure anisotropy, and the role of magnetic fields is essential for understanding how some stars explode while others form black holes [3, 4].

MHD, and its relativistic extension RMHD, provide the theoretical framework necessary to study these dynamics. The inclusion of magnetic fields introduces anisotropic stresses and wave modes (e.g., fast and slow magnetosonic waves, Alfvén waves) that significantly alter the nature of shock propagation compared to pure hydrodynamics [5, 6]. Moreover, RMHD is essential for modelling scenarios in which the fluid velocity approaches the speed of light, such as jets in gamma-ray bursts or shocks in relativistic outflows [7, 8].

To validate numerical RMHD codes and ensure physical accuracy, test problems with known solutions or expected qualitative behaviours are employed. Among these, the piston-driven shock tube problem stands out as a valuable benchmark. In this setup, a piston pushes into a medium, generating a shock front that evolves over time. This simple yet powerful model approximates several key features of CCSNe and other explosive phenomena: it contains sharp gradients, involves both compressive and expansive wave structures, and allows investigation of magnetic field amplification through compression.

In the present study, a one-dimensional piston-driven shock tube problem in both hydrodynamic and MHD contexts is developed and analysed using Python. As part of the validation strategy, the well-known Brio–Wu shock tube problem [9] is simulated, which serves as a classical MHD test case involving multiple wave structures and discontinuities. The model is implemented using the PLUTO code [10], an open-source computational fluid dynamics tool tailored for astrophysical flows. The objective is to evaluate the ability of the PLUTO code to handle RMHD shock structures and provide a basis for more complex, multidimensional simulations in astrophysics. By establishing the fidelity of numerical results against analytical expectations and known physical behaviours, the aim is to contribute to the development of effective computational frameworks for high-energy astrophysical modelling.

2 Theoretical and Computational Framework

This study aims to construct a robust piston-driven shock model capable of describing magnetised astrophysical plasmas in extreme conditions, such as those encountered in core-collapse supernovae (CCSNe). While classical test problems such as the Brio–Wu shock tube are invaluable not only for describing the HIC model but also for verifying numerical solvers, they do not reflect the physical mechanisms governing shock generation in realistic supernova environments.

To address this limitation, a piston-driven shock model was designed specifically tailored for RMHD conditions. However, given the inability to locate publicly available RMHD piston-driven shock models in open-source frameworks through online search, this work takes a stepwise approach: starting from classical hydrodynamics, analytical model is built and validated for piston-driven shocks, extend these to MHD (in theory and Python), and ultimately aim to develop a fully relativistic RMHD version compatible with the PLUTO code. This framework section lays out the theoretical foundation, starting from general conservation laws, then narrows down to the formulation required for numerical implementation, and finally explains the structure of our validation pipeline.

Conservation Laws in Relativistic Magnetohydrodynamics

The evolution of a magnetised relativistic fluid is governed by conservation of baryon number and energy-momentum [11, 5]:

$$\partial_\mu N^\mu = 0 \qquad \text{(Mass Conservation)} \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{(Energy-Momentum Conservation)} \tag{2}$$

Here, $N^\mu = \rho u^\mu$ is the rest-mass current, where ρ is the proper rest-mass density and u^μ the fluid four-velocity. The total stress-energy tensor includes both matter and electromagnetic contributions:

$$T^{\mu\nu} = (\rho h + p)u^\mu u^\nu + p g^{\mu\nu} - b^\mu b^\nu + \frac{1}{2} b^2 g^{\mu\nu}$$

where $h = 1 + \epsilon + p/\rho$ is the specific enthalpy, p is the thermal pressure, b^μ is the magnetic field four-vector in the fluid rest frame, and $g^{\mu\nu}$ is the spacetime metric. These equations reduce to their hydrodynamic or non-relativistic forms by neglecting appropriate terms. The inclusion of b^μ introduces anisotropic stresses and additional wave modes (e.g., Alfvén, fast/slow magnetosonic), making RMHD essential for modelling shock propagation in magnetised CCSNe scenarios.

Numerical Formulation

For computational applications, Eqs. (2)–(1) are written in conservative form:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \tag{3}$$

with:

$$\mathbf{U} = \begin{pmatrix} D \\ S^x \\ \tau \\ B^y \\ B^z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} D v^x \\ S^x v^x + p^* - (b^x)^2 \\ S^x - D v^x + p^* v^x - b^0 b^x \\ B^y v^x - B^x v^y \\ B^z v^x - B^x v^z \end{pmatrix}$$

Here:

$$D = \rho\gamma, \quad S^x = \rho h\gamma^2 v^x + b^2 v^x - b^0 b^x, \quad \tau = \rho h\gamma^2 - p + \frac{1}{2}b^2 - (b^0)^2 \tag{4}$$

D is the conserved rest-mass density, S^x is x-component of the total momentum density, τ is the total energy density minus rest-mass energy, $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor, $\vec{v}$ is the fluid velocity, and $p^* = p + \frac{1}{2}b^2$ is the total pressure. These expressions define the conserved quantities $\mathbf{U}$ and their fluxes $\mathbf{F}$ as functions of the primitive variables [7].

In this study, the mathematical framework outlined above is applied in two complementary computational settings. First, a stand-alone Python implementation of the 1D piston-driven shock problem is developed based on the conservation laws and shock conditions, allowing for the generation of controlled benchmark profiles under hydrodynamic and magnetohydrodynamic (MHD) conditions. Second, the PLUTO code is used to numerically solve the same equations using high-resolution shock-capturing schemes, including Riemann solvers. This dual-framework approach enables both analytical insight and robust numerical verification of MHD shock dynamics, serving as a foundation for future relativistic RMHD extensions.

Rankine–Hugoniot Jump Conditions for Shock Analysis

To compute analytical post-shock states for benchmark validation, the Rankine–Hugoniot jump conditions for relativistic magnetohydrodynamics (RMHD) are applied across the shock front [5].

$$[\![\rho\gamma\vec{v}]\!] = 0 \quad \text{(Mass Flux Continuity)} \tag{5a}$$

$$[\![(\rho h + b^2)\gamma^2 v^2 + p + \frac{1}{2}b^2 - (b^x)^2]\!] = 0 \quad \text{(Momentum Flux)} \tag{5b}$$

$$[\![(\rho h + b^2)\gamma^2 \vec{v} - b^0 b^x]\!] = 0 \quad \text{(Energy Flux)} \tag{5c}$$

These conditions enforce conservation of mass, momentum, and energy in the presence of both relativistic velocities and magnetic fields. The symbol $[\![\cdot]\!]$ denotes the jump in a quantity across the discontinuity (i.e., the difference between post-shock and pre-shock values). The fluid velocity $\vec{v}$ entering the flux terms is defined relative to the shock front. In practical implementation, the shock speed $\vec{v}_s$ is specified or iteratively adjusted in the laboratory frame, and the fluid velocities on either side of the shock are transformed into the shock rest frame via $\vec{v} = \vec{v}_{\text{fluid}} - \vec{v}_s$. The full RMHD jump conditions include contributions from the fluid pressure, specific enthalpy, and electromagnetic stresses via the magnetic field four-vector b^μ. Together with a closure relation such as the relativistic ideal gas equation of state, $p = (\Gamma - 1)\rho\epsilon$, the jump conditions form a non-linear algebraic system that can be solved numerically. The resulting post-shock profiles for pressure, density, velocity, and magnetic field serve as reference solutions against which RMHD code outputs can be validated.

Test Case Rationale and Model Setup

While the Brio–Wu shock tube is a classical and well-established RMHD test, it does not model how shocks are physically launched in CCSNe, however it can be used to model HIC environment. Therefore, for the purpose of this study, the Brio–Wu is retained solely to verify that our numerical setup within PLUTO correctly resolves known RMHD wave structures, such as discontinuities and slow/fast shocks. The core of this study is the piston-driven shock model. Since no off-the-shelf RMHD piston-driven shock models has been discovered in PLUTO or no readily accessible models could be located online within open-source numerical frameworks, the following staged development process is employed:

1. Start with a classical hydrodynamic piston-driven shock model to understand shock generation from a moving boundary.

2. Develop a full analytical solution in Python for this HD case, including post-shock profiles.

3. Extend the model by incorporating magnetic fields into the governing equations, yielding an MHD piston-driven shock model.

4. Construct the numerical solver in Python and validate it against the analytical jump conditions.

5. Finally, prepare a full RMHD implementation of the piston-driven shock model in PLUTO by translating and validating the extended framework.

At the current stage, the RMHD piston-driven shock model is still under development within Python. Hence, the RMHD piston-driven shock simulation results are excluded from this paper but retain both the Brio–Wu test for code validation and the HD piston-driven shock model for analytical benchmarking and model building. This approach reflects the incremental construction of a CCSNe-relevant piston-driven shock model with growing complexity: from HD to MHD to RMHD. This strategy also supports the long-term goal, which is to use the piston-driven RMHD model to simulate bounce-induced shock generation and propagation in magnetised core-collapse supernovae environments, where magnetic pressure and field topology critically shape the post-bounce dynamics.

3 Problem Set-Up

Two benchmark problems are considered in this study to evaluate the accuracy of numerical shock solvers and validate the analytical framework: the Brio–Wu shock tube model and a hydrodynamic (HD) piston-driven shock model. Both are implemented in one spatial dimension (1D) and provide complementary tests of shock capturing, wave structure resolution, and thermodynamic consistency.

Brio–Wu Shock Tube Model

The Brio–Wu shock tube [9] is a classical magnetohydrodynamic (MHD) test problem. It consists of a one-dimensional domain initially divided by a diaphragm separating two distinct fluid states. At time $t = 0.4$, the diaphragm is removed, allowing the fluids to interact. The resulting evolution produces a rich sequence of wave structures including fast and slow magnetosonic shocks, rarefaction waves, a contact discontinuity, and a rotational discontinuity [6, 5]. This test is widely used to verify the ability of numerical codes to resolve MHD wave modes and sharp discontinuities in magnetised plasmas. In this work, the Brio–Wu model is used specifically to validate the PLUTO code's RMHD solver and ensure it reproduces expected wave behaviour under known initial conditions.

Hydrodynamic Piston-Driven Shock Model

The piston-driven shock model represents a physically motivated setup for simulating shock formation through mechanical compression. In this configuration, a piston is initially positioned at location x_0 and begins moving into a stationary, un-magnetised fluid at time $t = 0.4$. The piston's motion imposes a discontinuity that generates a rightward-propagating shock wave and a leftward-expanding rarefaction fan [12]. This test emulates conditions relevant to core-collapse supernovae (CCSNe), where a bounce shock is launched into infalling stellar material. Unlike idealised symmetric setups, the piston-driven shock model allows for a controlled and directed shock propagation that mimics real astrophysical triggers. It is governed by the conservation of mass, momentum, and energy in a relativistic hydrodynamic (RHD) context, and admits analytical solutions through the Rankine–Hugoniot jump conditions. These solutions serve as reference profiles for benchmarking the numerical outputs of the PLUTO code [10].

Together, these two models offer both canonical and physically motivated frameworks for validating shock solvers in high-energy astrophysical environments. The Brio–Wu test affirms the code's ability to resolve multiple wave modes in magnetised settings, while the piston-driven shock model provides targeted control over shock strength and propagation geometry, forming a foundation for future 2D and 3D CCSNe simulations.

4 Results and Discussion

Brio–Wu Shock Tube Model

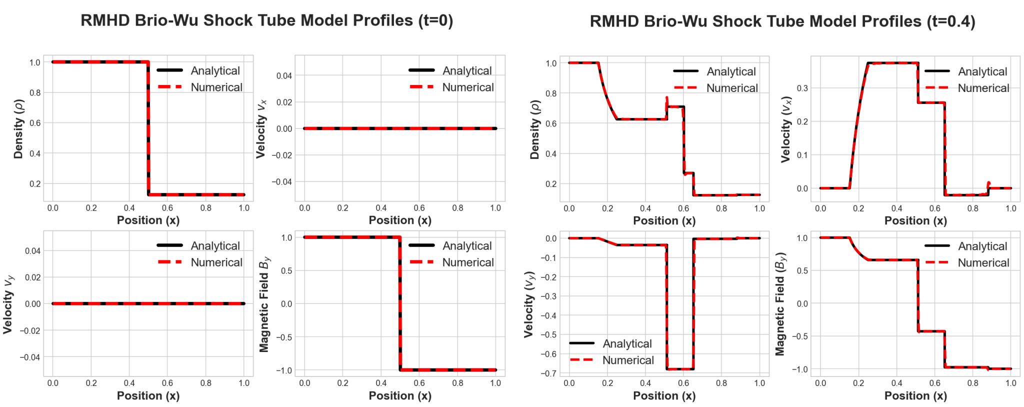

Figure 1: (*Left*) Initial conditions for the Brio–Wu RMHD shock tube at $t = 0.0$. (*Right*) Numerical profiles at $t = 0.4$ showing multiple wave structures.

Figure 1 presents the Brio–Wu RMHD shock tube test. The left panel shows the initial discontinuity in density, pressure, and magnetic field at $t = 0.0$, which initiates a complex wave interaction across the domain. By $t = 0.4$ (right panel), the simulation reproduces the full RMHD wave fan expected for this problem. The resulting wave structure includes a fast rarefaction ($x \approx 0.05$–0.25), a rotational discontinuity ($x \approx 0.35$), a contact and slow shock ($x \approx 0.45$–0.55), and a fast shock ($x \approx 0.65$). These features, including the flip in the transverse magnetic field B_y and stepwise changes in velocity and density, illustrate the solver's ability to accurately capture anisotropic MHD wave propagation.

These numerical results are in excellent agreement with the exact solutions presented by Rezzolla and Giacomazzo [13], as well as the numerical benchmarks from the PLUTO code [10], demonstrating that the implemented solver correctly reproduces the full RMHD wave structure for this standard test problem.

While this test remains a benchmark for assessing RMHD schemes, it has limited physical correspondence to realistic astrophysical systems. Therefore, an alternative piston-driven setup is developed to provide a more physically motivated analogue for validating MHD shocks in supernova environments.

Hydrodynamic Piston-Driven Shock Model

The hydrodynamic piston-driven shock test features a piston located at x_0, which begins moving into a stationary fluid at $t = 0.4$. The resulting shock propagates into the medium, and rarefaction waves trail behind, as illustrated in Figure 2.

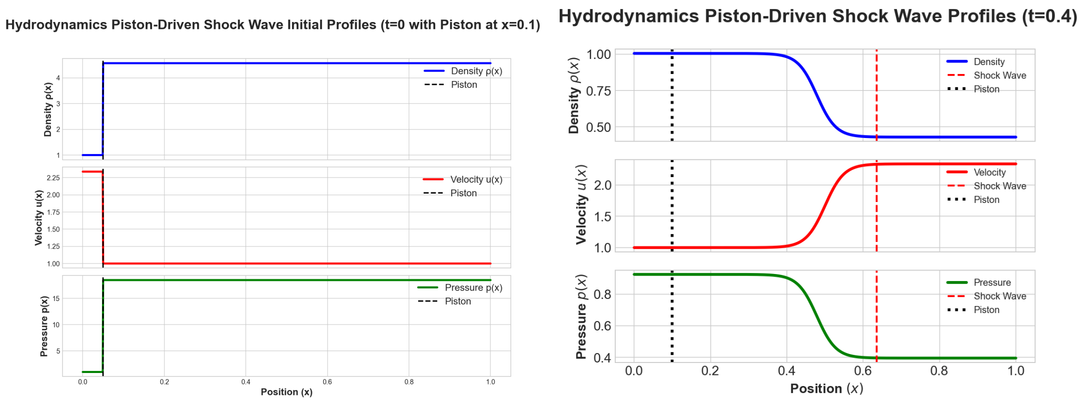

Figure 2: (*Left*) Initial conditions for the HD piston-driven shock model. (*Right*) Fluid variable profiles at $t = 0.4$ showing shock and rarefaction features.

The Python-based analytical solution shows a single, well-defined forward shock accompanied by a rarefaction region, consistent with expectations from the Rankine–Hugoniot jump conditions. These features are clearly resolved in the numerical profiles at $t = 0.4$, shown in the right panel of Figure 2. The HD piston-driven shock test also highlights the importance of resolution: non-physical oscillations observed behind the shock front in some simulations may indicate insufficient grid refinement. This serves as a reminder that stability and accuracy in compressible flow simulations are sensitive to spatial resolution and solver design.

Although this study omits the magnetic case for conciseness, the piston-driven shock model remains a versatile tool. Its extension to RMHD (planned in future work) would allow exploration of magnetic field amplification, pressure anisotropies, and more realistic CCSNe conditions. Even in the hydrodynamic limit, however, the piston-driven shock model offers critical insight into shock propagation and validation of high-resolution schemes such as those implemented in PLUTO. Moreover, the controlled piston-driven shock setup is better suited for adaptation to multi-dimensional supernova models. As such, it represents a necessary step toward full relativistic validation in more complex geometries.

5 Conclusion

This study implemented and validated two complementary one-dimensional test problems; the Brio–Wu shock tube and a hydrodynamic piston-driven shock model to evaluate the shock-capturing capabilities of numerical model. The Brio–Wu test confirmed the PLUTO code's ability to resolve complex RMHD wave structures, while the piston-driven model provided a physically motivated hydrodynamic setup representative of shock propagation in core-collapse supernovae.

By studying the analytical solutions derived from the Rankine–Hugoniot conditions, the piston-driven shock model was shown to be a reliable benchmark for building a foundation of the numerical model of the piston-driven shock model with accuracy in compressible flow regimes.

Future work will extend this framework to include magnetic fields and relativistic effects, enabling full RMHD benchmarking. This will enhance simulation fidelity for modelling magnetised shocks in high-energy astrophysical systems such as core-collapse supernovae.

Acknowledgements
The authors would like to thank the Centre for Broadband Communication team, Dr Zelalem Urgessa at Nelson Mandela University, Dr Isobel Kolbe at University of Witwatersrand, the Postgraduate Research Office at Nelson Mandela University, the National Research Foundation (NRF), and the National Institute for Theoretical and Computational Sciences (NITheCS) for support.

References

[1] S. Mrówczyński, "Physics at the rhic and lhc," *Acta Physica Polonica B*, vol. 48, no. 5, pp. 707–732, 2017.

[2] U. Heinz and R. Snellings, "Ultrarelativistic heavy-ion collisions: Collective dynamics and qcd matter at extreme temperatures," *Annual Review of Nuclear and Particle Science*, vol. 63, pp. 123–151, 2013.

[3] A. Burrows, "Colloquium: Perspectives on core-collapse supernova theory," *Reviews of Modern Physics*, vol. 85, no. 1, p. 245, 2013.

[4] K. Kotake, K. Sato, and K. Takahashi, "Core-collapse supernovae as gravitational wave sources," *Reports on Progress in Physics*, vol. 69, no. 4, p. 971, 2006.

[5] J. P. Goedbloed and S. Poedts, *Principles of Magnetohydrodynamics*. Cambridge University Press, 2004.

[6] B. Giacomazzo and L. Rezzolla, "The whisky code: a new numerical code for general relativistic magnetohydrodynamics," *Classical and Quantum Gravity*, vol. 24, no. 12, pp. S235–S258, 2007.

[7] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics*. Oxford University Press, 2013.

[8] J. M. Martí and E. Müller, "Recent developments in numerical relativistic hydrodynamics," *Living Reviews in Computational Astrophysics*, vol. 1, no. 1, p. 3, 2015.

[9] M. Brio and C. Wu, "An upwind differencing scheme for the equations of ideal magnetohydrodynamics," *Journal of Computational Physics*, vol. 75, no. 2, pp. 400–422, 1988.

[10] A. Mignone, G. Bodo, and S. e. a. Massaglia, "Pluto: a numerical code for computational astrophysics," *The Astrophysical Journal Supplement Series*, vol. 170, no. 1, p. 228, 2007.

[11] A. Anile, *Relativistic Fluids and Magneto-Fluids: With Applications in Astrophysics and Plasma Physics*. Cambridge University Press, 1989.

[12] Aerodynamics for Students, "Shock and expansion waves — gas dynamics," n.d., accessed: 31 July 2025. [Online]. Available: https://aerodynamics4students.com/gas-dynamics-and-supersonic-flow/gasdynamics_w.php?page=9a&

[13] B. Giacomazzo and L. Rezzolla, "The exact solution of the riemann problem in relativistic magnetohydrodynamics," *Journal of Fluid Mechanics*, vol. 562, pp. 223–259, 2006.